An Energy Saving Control Method of Robot Motions
based on Adaptive Stiffness Optimization
- Cases of Multi-Frequency Components -

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Abstract—This paper proposes an energy saving control method of periodic motions composed of multi-frequency components based on adaptive stiffness optimization for mechanical systems. In the case of sinusoidal motions, the concept of resonance can be easily applied to determine an optimal stiffness. This paper tries to extend resonance to deal with periodic motions composed of multi-frequency components. For this purpose, we define a cost function, which represents an amount of actuator torque. Next, an optimal stiffness is defined as the stiffness that minimizes the cost function. The relationship between the optimal stiffness and the desired motions is analytically derived. Based on the preparations, we propose two types of controller. These controllers generate the desired motions using the least amount of actuator torque as possible by optimizing the stiffness adaptively. The controllers work well without using parameters of the objective systems. Stability and effectiveness of the stiffness optimization are proved mathematically. Simulation results demonstrate the effectiveness of the proposed method.

I. INTRODUCTION

The importance of energy saving seems to increase dramatically in the near future due to environmental problems. For mechatronics systems, energy saving is one of fundamental requirements. Mobile robots require energy saving techniques, because they have to equip energy sources in themselves. From a scientific viewpoint, energy saving techniques are also interesting because it is expected that animals and human beings save energy as much as possible.

In the robotics field, control methods based on dynamic characteristics of objective systems, such as passive walking robots, have been proposed so far [1][2][3][4]. They utilize energy storage and release of gravitational effects or mechanical elastic elements to generate periodic motions without using torque of actuators. Living things also seem to utilize passive elements, such as tendons and muscles, to save energy.

Fujimoto et al. proposed a control method, which generates periodic motions of hopping robots without using actuator torque in steady states [5]. However, in this method, optimality can be available only in the case of a few specific motions.

On the other hand, Ozawa et al. proposed a control method, which generates desired sinusoidal motions without energy supply in steady states [6]. This method optimizes stiffness of a certain type of objective system adaptively without using parameters of the objective systems. The optimization is based on anti-resonance. The authors also proposed power assist systems, which simultaneously realize amplification of its operator's torque and optimization of stiffness to minimize actuator torque [7][8].

However, in the previously proposed control methods utilizing the stiffness optimization, there are the following limitations.

1) Dynamics of the objective systems is linear.
2) Degree-of-freedom is limited to one.
3) Desired motions must be sinusoidal.

These limitations seem to come from direct use of anti-resonance or resonance, because (anti) resonance is the concept for one degree-of-freedom linear systems and sinusoidal motions originally.

Our goal is to develop energy saving control methods for various robotic systems that perform various tasks as shown in Fig.1. For example, when multi-joint robots perform pick-and-place tasks, dynamics of the robots becomes nonlinear and multi degree-of-freedom. Desired motions can not become sinusoidal, because the motions should have some stopping periods to pick and to place some objects. To deal with the problems, the resonance phenomenon should be effectively extended for the purpose of energy saving.

We have already proposed an energy saving control method utilizing adaptive stiffness optimization for multi-joint robots [9]. In this method, resonance is extended to deal with robot dynamics that has nonlinearity and multi degree-of-freedom. However, desired motions of the our previous method are still limited to sinusoidal motions.

This paper tries to solve the problem of the desired motions. Namely, desired motions in this paper are composed of multi-frequency components. To avoid complexity on the analysis, dynamics of the objective systems is assumed to be

Fig. 1. Robots Performing Periodic Tasks
linear.

Next, we define an optimal stiffness and introduce a cost function. The cost function represents an amount of actuator torque needed to generate the desired motions. An optimal stiffness is defined as a stiffness that minimizes the cost function. When we use a certain type of DC motor as the actuator, this minimization directly means minimization of energy consumption. We analytically derive relationship between the desired motions and the optimal stiffness.

Basing on the preparations, we propose two types of controller. Both controllers generate the desired motions using the least amount of actuator torque as possible by optimizing the stiffness, and they do not use physical parameters of the objective systems.

II. PROBLEM FORMULATION

This section formulates the problem of this study.

A. Dynamics of Objective System

The objective system is a mechanical system, which has an adjustable stiffness device, one degree-of-freedom, and linear dynamics as shown in Fig.2.

\[
m \ddot{q}(t) + d \dot{q}(t) + k \dot{q}(t) = \tau(t) - k(t)q(t),
\]

where \(m\), \(d\), \(k\) are coefficients of inertia, viscosity and stiffness of the mechanical system, respectively. \(q\) and \(\tau\) are an angle and torque, respectively. \(k\) is stiffness, which can be adjusted mechanically. In this paper, all variables and constants are scalar.

Many researchers have proposed structures to realize stiffness adjustment of mechanical devices [10]. Hence, some small and lightweight adjustable stiffness devices, which have large adjustable region, are available.

B. Desired Motion

The desired motion \(q_d\) is assumed to be a periodic motion, which is composed of \(n\) \((n \geq 2)\) frequency components

\[
q_d(t) = \sum_{i=1}^{n} a_i \sin(\omega_i t + \phi_i),
\]

where \(a_i\), \(\omega_i\), \(\phi_i\) are amplitude, angular frequency, and phase of each component. The cycle of the desired motion is \(T = \frac{2\pi}{\omega_i}\).

Then, the desired motion itself \(q_d\), the first time derivative \(\dot{q}_d\) and the second time derivative \(\ddot{q}_d\) are clearly finite.

\[
|q_d(t)| < c_1, \quad |\dot{q}_d(t)| < c_2, \quad |\ddot{q}_d(t)| < c_3
\]

A. Cost Function

It is well-known that arbitrary motions can be represented by sum of sinusoidal motions. Therefore, the equation (2) can represent motions of various tasks, such as pick-and-place tasks.

C. Control Objective

We define two control objectives. One is to achieve tracking control \(q(t) \rightarrow q_d(t)\). The other is to optimize the stiffness \(k\) to reduce the actuator torque \(\tau\) as much as possible.

Some additional conditions are discussed in detail in the section IV.

III. OPTIMAL STIFFNESS

This section describes the optimal stiffness in detail.

A. Cost Function

Here, we discuss a cost function to evaluate efficiency, 1) Evaluation from energy: We consider a cost function \(J_e\), which evaluates energy supplied from an electrical current source to the objective system. Necessary energy from the electrical current source to generate the desired motion can be written by

\[
J_e = \int_0^T v_d(t) i_d(t)dt,
\]

where \(v_d\) and \(i_d\) are voltage and current needed to generate the desired motion, respectively. In the case of DC motors, modeling among the voltage \(v_d(t)\), the current \(i_d(t)\) and the motion \(q_d(t)\) is given by

\[
v_d(t) = L \dot{i}_d(t) + R i_d(t) + k_i \dot{q}_d(t),
\]

where \(L\) is an inductance, \(R\) is a resistance, and the \(k_i\) is a torque constant.

This type of DC motor exerts torque proportionally to the current \(k_i \dot{i}_d = \tau_d\). The \(\tau_d\) is necessary torque to generate the desired motion. The necessary torque \(\tau_d\) can be calculated from substituting the desired motion \(q_d(t)\) and a constant stiffness \(k_s\) into \(q(t)\) and \(k(t)\) of the dynamics (1), respectively.

\[
\tau_d(k_s, t) = m \ddot{q}_d(t) + d \dot{q}_d(t) + k_s q_d(t) + k_s q_d(t)
\]

Then, the energy consumption in one cycle \(J_e\) can be concretely calculated as follows:

\[
J_e = E_c(T) - E_c(0) + R \int_0^T \dot{q}_d^2 dt + \int_0^T k_s \dot{i}_d \dot{q}_d dt
\]

\[
= R \int_0^T \dot{i}_d^2 dt + E_m(T) - E_m(0) + d \int_0^T \dot{q}_d^2 dt
\]

\[
= R \int_0^T \dot{i}_d^2 dt + d \int_0^T \dot{q}_d^2 dt,
\]

where \(E_c(t) = \frac{1}{2} L \dot{i}_d(t)^2\), and \(E_m(t) = \frac{1}{2} m \dot{q}_d(t)^2 + \frac{1}{2} (k_s + k_s) q_d(t)^2\). Because the desired motion is periodic, \(E_c(T) - E_c(0)\) and \(E_m(T) - E_m(0)\) become zero.

The equation (7) means that maximization of the energy efficiency is achieved by minimizing the norm of the necessary torque \(\int_0^T \dot{q}_d^2 dt\).
In the cases of other actuators, such as AC motors, a relationship between current and the torque also tends to have linear characteristic. In such cases, to minimize the necessary torque \( \tau_d \) is also effective for energy saving.

2) Definition of cost function \( J(k_s) \): Based on above discussion, we define a cost function \( J(k_s) \) that our control method tries to minimize. The \( J(k_s) \) is the norm of the torque needed to generate the desired motion.

\[
J(k_s) = \int_0^T \tau_d(k_s, t)^2 dt
\]  
\( (8) \)

B. Definition of Optimal Stiffness \( k_{opt} \)

The optimal stiffness \( k_{opt} \) is defined as the stiffness that minimizes the cost function \( J(k_s) \). Hence, the minimum of the \( J(k_s) \) is defined as \( J_{opt} \) as follows:

\[
J_{opt} = J(k_{opt}) = \min_{k_s} J(k_s).
\]  
\( (9) \)

Optimal actuator torque \( \tau_d(k_{opt}, t) \) is also defined as \( \tau_{dopt}(t) = \tau_d(k_{opt}, t) \).

C. Relationship between Desired Motion \( q_d \) and Optimal Stiffness \( k_{opt} \)

Here, we clarify the relationship between the desired motion \( q_d \) and the optimal stiffness \( k_{opt} \).

At first, we calculate the torque needed to generate the desired motion by substituting the parameters of the desired motion \( q_d \) of the equation (2) into the equation (6).

\[
\tau_d(k_s, t) = \sum_{i=1}^{n} a_i \left\{ -i^2 m \omega^2 \sin(i \omega t + \phi_i) + i k_s \omega \cos(i \omega t + \phi_i) + (k_h + k_s) \sin(i \omega t + \phi_i) \right\}
\]  
\( (10) \)

Next, we substitute the necessary torque of the equation (10) into the cost function \( J(k_s) \).

\[
J(k_s) = \int_0^T \left[ \sum_{i=1}^{n} a_i \left\{ -i^2 m \omega^2 - k_h - k_s \sin(i \omega t + \phi_i) + i \omega \cos(i \omega t + \phi_i) \right\}\right]^2 dt
\]  
\( (11) \)

By using orthogonality of trigonometric functions, the definite integral of product of \( \sin(i \omega t + \phi_i) \) and \( \cos(j \omega t + \phi_j) \) from 0 to \( T \) becomes 0. Similarly, definite integral of product of \( \sin(i \omega t + \phi_i) \) and \( \sin(j \omega t + \phi_j)(j \neq i) \), and that of \( \cos(i \omega t + \phi_i) \) and \( \cos(j \omega t + \phi_j)(j \neq i) \) from 0 to \( T \) become 0. Therefore, the equation (11) becomes

\[
J(k_s) = \int_0^T \sum_{i=1}^{n} a_i^2 \left\{ (i^2 m \omega^2 - k_h - k_s)^2 \sin(i \omega t + \phi_i)^2 + i^2 \omega^2 \sin(i \omega t + \phi_i)^2 \right\} dt
\]  
\( (12) \)

\[
= \frac{T}{2} \sum_{i=1}^{n} a_i^2 \left\{ (i^2 m \omega^2 - k_h - k_s)^2 + i^2 \omega^2 \right\} \quad \text{(13)}
\]

We can obtain the following equation by partially differentiating the equation (13) by \( k_s \).

\[
\frac{\partial J(k_s)}{\partial k_s} = -T \sum_{i=1}^{n} a_i^2 \left( i^2 m \omega^2 - k_h \right) + k_s T \sum_{i=1}^{n} a_i^2 \quad \text{(14)}
\]

The equation (14) means that \( J(k_s) \) is a concave quadratic function of \( k_s \).

At last, the relationship between the optimal stiffness \( k_{opt} \) and the desired motion can be derived as follows:

\[
k_{opt} = \frac{\sum_{i=1}^{n} a_i^2 m \omega^2 - k_h}{\sum_{i=1}^{n} a_i^2}.
\]  
\( (15) \)

IV. CLASSIFICATION OF SITUATION

To utilize our control method in some situations, this paper considers the following two cases.

1) case 1

In this case, the desired motions with multi-frequency components are given in advance but the parameters of the objective systems are not given. In many industrial applications, the desired motions are determined in advance, even if the desired motions are changed depending on the tasks.

2) case 2

The second case treats that neither the information of the desired motions nor the system parameter is given. For example, in the cases of power assist systems, human operators produce desired motions online or the information of the desired motion can not be available in advance [8]. In this case, we should assume that the human operator generates periodic motions but the parameters of the periodic motions are unknown.

V. CONTROL METHOD I (USING ADAPTIVE FEEDFOWARD COMPENSATION)

This section proposes a control method to deal with the case 1 of the previous section.

Even though the dynamics of the objective systems is linear, we need a nonlinear scheme to achieve the control objectives due to the requirement of the stiffness optimization. The control method in this section utilizes adaptive feedforward compensation as the nonlinear scheme.

A. Controller

We propose the following controller

\[
\tau = \dot{m}q + \dot{\dot{q}} + i a_d \omega^2 q - k_p \Delta q - k_v \Delta \dot{q}
\]  
\( (16) \)

\[
\dot{\dot{q}} = -\gamma_1 (\dot{q} + a_d \omega^2 q) \Delta \dot{q}
\]  
\( (17) \)

\[
\dot{\dot{q}} = -\gamma_d \dot{q} \Delta \dot{q}
\]  
\( (18) \)

\[
\dot{\dot{k}} = \gamma_k \Delta \dot{q}
\]  
\( (19) \)

where \( a_d = \sum_{i=1}^{n} a_i^2 \), \( \Delta q = q - q_d \), \( k_p \), \( k_v \) are feedback gains, and \( \gamma_1, \gamma_d, \gamma_k \) are adaptive gains. This controller does not use any parameters of the objective system.

The equations from (16) to (19) are similar to usual adaptive controllers. An original structure of the proposed controller is to use the term \( i a_d \omega^2 \) in the equation (16) as an estimate value of the stiffness of the mechanical system \( k_h \). This structure enables that the stiffness \( k \) becomes optimal one \( k_{opt} \) in equilibria.
B. Stability

Stability and convergence of the parameters \( \hat{m}, \hat{d}, k \) are proved by the following Lyapunov function \( V \):

\[
2V = m\Delta q^2 + \left( k_h + k_{opt} + k_p \right) \Delta q^2 + \gamma_1^2 \Delta m^2 + \gamma_d^2 \Delta d^2 + \gamma_k^2 \Delta k^2
\]

\[
\dot{V} = -(d + k_v)\Delta q^2,
\]

where \( \Delta m = \hat{m} - m, \Delta d = \hat{d} - d \), and \( \Delta k = k - k_{opt} \).

LaSalle’s invariant theorem guarantees convergence of \( \Delta \dot{q} \) to 0 as \( t \to \infty \). Then, it is clear that \( \Delta \dot{q} \to 0, \Delta q \to \Delta q_e \) as \( t \to \infty \), where \( \Delta q_e \) is an integral constant. From the equations from (17) to (19), the parameters \( \hat{m}, \hat{d}, k \) become constant, because \( \Delta \dot{q} \) converges to 0 as \( t \to \infty \). Thus, the dynamics (1) and the controller (16) bring about the following equation when \( t \to \infty \):

\[
(m - \hat{m})\ddot{q} + (d - \hat{d})\dot{q} + (k_h + k - ma_\omega^2)q = 0
\]

(22)

The equation (22) is satisfied only when \( \Delta q_e = 0, \hat{m} = m, \hat{d} = d, k = k_{opt} \).

Above discussion concludes that the parameters \( k, \hat{m}, \hat{d} \) converge to the desired ones \( k \to k_{opt}, \Delta \dot{q} \to 0, \hat{m} \to m, \hat{d} \to d \), as \( t \to \infty \). Then, we can obtain the relationship \( \tau \to \tau_{opt} \) by substituting the converged parameters into the equation (16).

Hence, it is proved that the proposed controller of the equations from (16) to (19) realizes the control objectives without using any parameters of the objective system.

VI. CONTROL METHOD 2 (WITHOUT USING ADAPTIVE FEEDFOWARD COMPENSATION)

This section proposes a controller to deal with the case 2 of the section IV. This controller does not use the adaptive feedforward compensation. In this case, the controller uses only instantaneous information of \( q, \dot{q} \) and \( q_d, \dot{q}_d \).

The controller of the equations from (16) to (19) in the previous section is based on the analytical relationship between the optimal stiffness \( k \) and the parameters \( m, \omega, k_h \), which is derived from the equation (15). However, if we will extend the concept of our control method to nonlinear systems, the relationship like the equation (15) seems not to be solved analytically. In such case, the proposed controller in the previous section can not be extended to the systems.

A. Controller Design

The controller is given by

\[
\tau = -k_v\Delta \dot{q} - k_p\Delta q
\]

\[
\dot{k} = \gamma_k q(\Delta \dot{q} + \alpha \Delta q),
\]

where \( k_v, k_p \) are feedback gains, \( \Delta q = q - q_d \), the \( \alpha \) is a positive constant, and the \( \gamma_k \) is an adaptive gain. The feedback gain \( k_p \) is set \( k_p = ak_v \).

This controller does not use the parameters of the objective system, and the information of the desired motion \( a_i (i = 1, 2, \cdots n) \), \( \omega \).

B. Stability

The following scalar function \( V_2(t) \) guarantees stability of the controlled system.

\[
V_2(t) = V_a(t) + c_4 V_b(t)
\]

(25)

where \( c_4, c_5 \) are constants to prove the stability.

1) Assumption: To prove the stability, we set the following two assumptions.

- **assumption 1** The PE condition \( \int_{t}^{t+T} q^2 dt > c_{pe} \) is satisfied, where the \( c_{pe} \) is a positive constant.

- **assumption 2** The trajectory tracking errors \( \Delta \dot{q}, \Delta q \) are in the certain region \( \Delta q^2 < c_6, \Delta q^2 < c_7 \). The \( c_6, c_7 \) are positive constants.

Under these assumptions, this paper prove the boundedness of the scalar function \( V_2(t) \). Since this function \( V_2(t) \) includes the term \( \Delta k^2 \), we can prove the boundedness of \( \Delta k^2 \) by proving the boundedness of \( V_2(t) \). Note that these constants \( c_6, c_7 \) can be large by choosing large feedback gain \( k_v \).

2) In the case of \( V_2(t) > V_{2min} \) during a cycle: At first, we consider the case that \( V_2(t) \) of the equation (25) is larger than a certain positive constant \( V_{2min} \) during a cycle. In this case, \( V_2(t) \) will decrease after the cycle. Then we have

\[
V_2(t + T) < c_8 V_2(t),
\]

(28)

where \( c_8 \) is a constant from 0 to 1.

Detail of the derivation of the inequality (28) from the equation (25) is written in the appendix A.

3) Other case: Next, we consider other case. In this case, there are some period that \( V_2(t) \) is smaller than \( V_{2min} \) in a cycle. Then, it is possible that \( V_2(t) \) will increase after the cycle. However, \( V_2(t + T) \) won’t be larger than a constant, because there is a period that \( V_2(t) \) is smaller than \( V_{2min} \), and ratio of increase of \( V_2(t) \) is limited. The appendix B shows the ratio of the increase.

4) Convergence of \( V_2(t) \): Above discussion concludes that \( V_2(t) \) converges to a certain region, and the stability is proved.

C. Effect of Stiffness Adjustment

To prove effect of the stiffness adjustment, we consider the following scalar function \( V_{a2} \).

\[
V_{a2} = V_a - \frac{1}{2} (k_h + k_{opt}) \Delta q^2
\]

(29)

\[
\dot{V}_{a2} = -(d - ma_\omega) \Delta q^2 + \gamma_k \Delta q (\Delta \dot{q} + \alpha \Delta q) + (\Delta q + \alpha \Delta q)(\tau - \tau_{opt})
\]

(30)

\[
2k_h \dot{V}_{a2} = -2k_h(d - ma_\omega) \Delta q^2 - 2k_a(\gamma_k k_h + k_{opt}) \Delta q^2 - \tau^2 - (\tau - \tau_{opt})^2 + \tau_{opt}^2
\]

We evaluate the effect of the stiffness adjustment by using the average of the terms of the equation (31) in a cycle. The
terms of the equation (31) in a cycle can be calculated by integrating the equation (31) from $t$ to $t + T$ as follows:

$$2k_c \{ V_{a2}(t + T) - V_{a2}(t) \}$$

$$= \int_t^{t+T} \left\{ -2k_c(d - ma)\Delta q^2 - 2k_c\alpha(k_h + k_{opt})\Delta q^2 - \tau^2 - (\tau - \tau_{dopt})^2 + \tau_{dopt}^2 \right\} dt. \quad (32)$$

Next, the average of the equation (32) can be calculated by substituting $0, T, 2T, \cdots, hT$ into $t$ in order. Then, we can calculate sum of $h$ cycles of them. The average can be calculated by dividing the sum by $h$, and substituting $\infty$ into $h$ as follows:

$$\frac{1}{h} \int_0^{hT} \left\{ \tau^2 + (\tau - \tau_{dopt})^2 + c_{24}\Delta q^2 + c_{25}\Delta q^2 \right\} dt$$

$$= J_{opt}, \quad (33)$$

where $c_{24} = 2k_c(d - ma)$, $c_{25} = 2k_c\alpha(k_h + k_{opt})$. We used the boundedness of the $V_{a2}$.

The the equation (33) represents that the amount of the actuator torque $\tau$ in a cycle is smaller than $J_{opt}$. Therefore, if the tracking errors $\Delta q^2$, $\Delta q^2$ are small, the actuator torque $\tau$ will be close to $\tau_{dopt}$. This means that accurate tracking control will be achieved by the nearly smallest torque. This situation can be achieved by selecting large feedback gains $k_c$, $k_p$. Even if the tracking errors are large, the actuator torque will be small with the increase of the tracking errors.

Above discussion proves that the proposed controller generates the desired motions using the least amount of actuator torque as possible by the stiffness adjustment.

VII. SIMULATION

We conducted numerical simulations to show the validity of the proposed controller.

In this paper, the simulation results of the control method 1 are omitted, because the first controller could achieve the perfect tracking control, and the stiffness could converge to the optimal one $k_{opt}$. This characteristic is the same as the mathematical proof of the section V. Hence, we show the simulation results of the second controller (without using adaptive feedfoward compensation) in the following way.

A. Condition

Dynamics of the objective system is assumed to be the equation (1), and physical parameters are set to $m = 0.5$, $d = 0.2$, $k_h = 2.0$.

The controller is the equation (23) and (24), and gains are set to $k_c = 100.0$, $c = 2.0$, $k_p = 200.0$, $\gamma_k = 60.0$.

Two types of desired motion are set as follows. The first desired motion is designed using the equation (2), and the parameters are set to $n = 2$, $a_1 = 1.0$, $a_2 = 0.5$, $\phi_1 = 0.0$, $\phi_2 = 0.3\pi$ while $0 \leq t < 20$, and $n = 3$, $a_1 = 0.5$, $a_2 = 1.0$, $a_3 = 0.5$, $\phi_1 = 0$, $\phi_2 = 1.5\pi$, $\phi_3 = 0.5\pi$ while $20 \leq t < 40$. We call the motion "desired motion A". This motion is designed to demonstrate the performance of the proposed controller.

The second desired motion is designed to consider a more concrete task. When industrial robots perform periodic tasks, they usually have some periods of stopping to pick up some objects or to place some objects. Therefore, the second desired motion is set to the following equations. $q_d(t) = a \sin \left( \frac{4}{5} \pi t \right)$ while $0 < t < \frac{1}{5}T$, $q_d(t) = a \sin \left( \frac{4}{5} \pi t \right)$ while $\frac{1}{5}T < t < \frac{2}{5}T$, $q_d(t) = a \sin \left( \frac{4}{5} \pi t \right)$ while $\frac{2}{5}T < t < \frac{3}{5}T$, $q_d(t) = a \sin \left( \frac{4}{5} \pi t \right)$ while $\frac{3}{5}T < t < T$. The period of the cycle $T$ is set to $T = 1.0$ while $0 < t < 20$, and the amplitude $a$ is set to $1.0$. We call the second desired motion "desired motion B".

B. Result

The simulation results of the desired motion A and the desired motion B are shown in Fig.3 and Fig.4, respectively.

In the case of the desired motion A, the angle $q$ almost converged to the desired one $q_d$ as shown in Fig.3(a). The stiffness $k$ also almost converged to the optimal one.
$k_{opt}$ as seen in Fig.3(b). Fig.3(c) shows that the optimal torque $\tau_{dopt} = \tau_d(k_{opt})$ becomes smaller than the necessary torque $\tau_d(0)$. This means that the torque of stiffness $k_{opt}q$ effectively reduced the necessary torque. In fact, the norm of the torque $\int_0^{20} \tau_{dopt}^2 \, dt$ becomes as small as 39% of the norm of the torque $\int_0^{20} \tau_d(0)^2 \, dt$ while $0 \leq t < 20$, and the ratio $\int_0^{20} \tau_{dopt}(0)^2 \, dt$ becomes 24% while $20 \leq t < 40$. The reason why the ratio is different (39% and 24%) is that the necessary torque $\tau_d(0)$ in the last 20[s] ($20 < t \leq 40$) can be compensated by the torque of the stiffness $kq$ more than the necessary torque $\tau_d(0)$ in the first 20[s] ($0 \leq t < 20$). The actuator torque $\tau$ almost converged to the optimal one $\tau_{dopt}$ as shown in Fig.3(d). Above results mean that the proposed controller cloud optimize the stiffness adaptively, and only small torque was needed to generate the desired motion.

In the case of the desired motion B, the variables $q$ and $k$ almost converged to the desired ones as shown in Fig.4(a) and Fig.4(b) respectively. In this case, $\int_0^T \tau_{dopt}^2 \, dt$ becomes 55% of $\int_0^T \tau_d(0)^2 \, dt$. As shown in Fig.4(a), the robot stopped in some periods. These results show that the proposed control method is effective even periodic motions have some periods of stopping like industrial robots.

VIII. CONCLUSIONS

This paper proposed an energy saving control method based on stiffness adjustment. Desired motions in this study have multi-frequency components. The proposed control method optimizes the stiffness adaptively to generate the desired motions using the least actuator torque as possible. This method works well without using parameters of the objective system.

This paper also analyzed an optimal stiffness in the case of the multi-frequency components. We introduced a cost function to evaluate the amount of the actuator torque to generate the desired motions. The optimal stiffness minimizes the cost function. As the results of the analysis, some characteristics of the optimal stiffness are clarified, and these characteristics are not included in the concept of conventional resonance.

The simulation results demonstrated effectiveness of the proposed method. These results showed that the stiffness optimization effectively reduced actuator torque while generating a desired motion, even when the desired motion included multi-frequency components.

In the future, we will extend this method to solve the problem of multi degree-of-freedom with multi-frequency components. We expect that the proposed concept is effective for such problem. Especially, the controller 2 of the section VI is a good candidate.

REFERENCES


APPENDIX

A. Derivation of Inequality (28) from Equation (25)

At first, we derive the time derivative of $V_2(t)$ of the equation (25). The time derivative of $V_a$ and $V_b$ of the equation (26) and (27) are as follow:

\[
\dot{V}_a = -(d + k_o)\Delta q^2 - \alpha (k \Delta q + k_{opt} + k_p)\Delta q^2 \\
+ (\Delta \dot{q} + \Delta q)\tau_{dopt} \\
\leq -(d + k_o)\Delta q^2 - \alpha (k \Delta q + k_{opt} + k_p - c_9)\Delta q^2 \\
+ \frac{1}{4c_9}\tau_{dopt}^2 \\
\dot{V}_b = -c_5q^2\Delta k^2 - m(d + k_o - c_5\gamma_k)q^2\Delta q^2 \\
+ (c_3\dot{m}q - mq^2)\Delta \dot{q}\Delta k \\
- c_5(k \Delta q + k_{opt} + k_p - c_5\alpha\gamma_k)q\Delta q\Delta k \\
- m(k \Delta q + k_{opt} + k_p - c_5\alpha\gamma_k)q^2\Delta \dot{q}\Delta q \\
- q\tau_{dopt}(mq\Delta \dot{q} + c_5\Delta k) + m^2q\Delta \dot{q}^2,
\]

where $c_9$ is a constant introduced for the theoretical discussion.

The inequalities $|q| < c_6 + c_1$, $|\dot{q}| < c_7 + c_2$ are satisfied by the assumption 2 and the characteristic of the desired motion. Hence, by using the assumption 2 again and the equation (35), we have

\[
\dot{V}_b \leq -c_{10}q^2\Delta k^2 + c_{11}\Delta k^2 + c_{12}\Delta q^2 \\
+ c_{13}\Delta \dot{q}^2 + c_{14}\tau_{dopt}^2,
\]

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where $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are positive constants.

Therefore, by adequate choice of $c_4$, the inequality (34) and the inequality (36), we have

$$
\dot{V}_2(t) \leq -c_{19} \Delta k^2 + c_{11} \Delta \dot{q}^2 - c_{19} \Delta \dot{q}^2
$$

$$
\leq -(c_{19} q^2 - c_{19}) V_2(t) + c_{17} \Delta q_{\text{opt}}^2.
$$

(37)

where $c_{15}, c_{16}, c_{17}, c_{19}$ are positive constants.

1) Relationship between $V_2(t)$ and $V_2(t + T)$: We can derive the following relationship between $V_2(t)$ and $V_2(t + T)$ by dividing the inequality (37) by $V_2(t)$, and integrating it from $t$ to $t + T$.

$$
\log V_2(t + T) - \log V_2(t)
\leq -c_{18} \int_t^{t+T} q^2 dt + c_{19} T + \int_t^{t+T} \frac{c_{17} \Delta q_{\text{opt}}^2}{V_2(t)} dt
$$

$$
= c_{20}
\therefore V_2(t + T) \leq e^{c_{20}} V_2(t)
$$

(38)

(39)

Therefore, if $c_{20}$ is negative $c_{20} < 0$, $V_2(t + T)$ will be smaller than $V_2(t)$ and the inequality (28) will be satisfied.

2) Condition of $c_{20} < 0$: Next, we consider the condition of $c_{20} < 0$. We can calculate the following inequality of $c_{20}$ from the inequality (38) by using the PE condition of the assumption 1 and the condition of $V_2(t) > V_{2\text{min}}$.

$$
c_{20} \leq -c_{18} \Delta q_{\text{pe}} + c_{19} T + \frac{c_{17} \Delta q_{\text{opt}}}{V_{2\text{min}}}
$$

(40)

Therefore, concrete $V_{2\text{min}}$, which results in the inequality $c_{20} < 0$, is given by

$$
V_{2\text{min}} \geq \frac{c_{17} \Delta q_{\text{opt}}}{c_{18} \Delta q_{\text{pe}} - c_{19} T}.
$$

(41)

The right hand side of the inequality (41) should be positive, because $V_{2\text{min}}$ is defined as a positive constant.

3) Condition that right hand side of inequality (41) will be positive: If $c_{10}$ is small enough, the right hand side of the inequality (41) will be positive. We can select small $c_{19}$ if $c_{11}$ of the inequality (36) is small. Therefore, we consider the derivation of the inequality (36) from the equation (35).

In the equation (35), the terms, which effect $c_{11}$, are the cross terms between $\Delta k$ and other terms, such as $\Delta k \Delta \dot{q}$. Hence, by selecting large $c_{12}, c_{13}, c_{14}$ of the inequality (36), $c_{11}$ can be small arbitrarily.

If $c_{12}, c_{13}, c_{14}$ become large, $c_4$ should be small due to the procedure of calculating the inequality (37) from the inequality (36). Then, $V_{2\text{min}}$ of the inequality (41) will be large because $c_{18}$ and $c_{19}$ should be large due to the small $c_4$. However, this problem can be solved by selecting large feedback gains $k_v, k_q$ that enables to select large $c_4$.

Therefore, it is proved that the inequality (28) is satisfied.

B. Maximum Increase of Scalar Function $V_2(t)$ in A Cycle

$$
\dot{V}_2(t) \leq c_{19} V_2(t) + c_{23}
$$

(42)

where $c_{23}$ is a positive constant.

From the inequality (42), $V_2(t + T)$ satisfies the following relationship.

$$
V_2(t + T) \leq e^{c_{19} T} V_2(t) + \frac{c_{23} \Delta q_{\text{opt}}}{c_{19} (e^{c_{19} T} - 1)}
$$

(43)

Hence, by introducing some constants $c_{21} = e^{c_{19} T} - 1, c_{22} = \frac{c_{23} \Delta q_{\text{opt}}}{c_{19} (e^{c_{19} T} - 1)}$, it is proved that the increase of $V_2(t)$ in a cycle is always smaller than $c_{21} V_2(t) + c_{22}$.